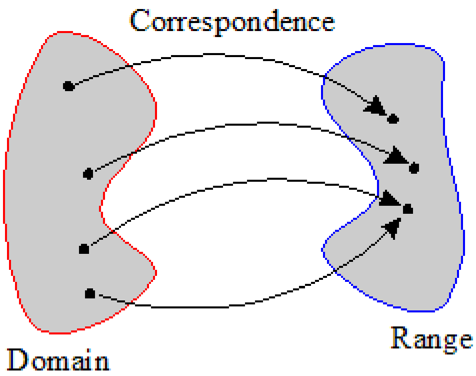
Fundamentals of Calculus

I. Theoretical Background

1. Function.

1.1 A **function** is a **correspondence** or mapping from a first set of numbers, called the **domain** of the function, to a second set of numbers, called the **range** of the function, such that for each member of the domain there is **exactly one** member of the range.



The concept of function embodies the ideas of:

a) Causal relationships between natural phenomena (known as regularities)

b) Changing over time or motion which are observed in nature (known as processes).

1) **Linear functions**. These are functions of the form**:** $\boldsymbol{y = a\,x + b}$,

where a and b are constants. A typical use for linear functions is converting from one quantity or set of units to another. Graphs of these functions are straight lines. a is the slope and b is the y intercept. If a is positive then the line rises to the right and if a is negative then the line falls to the right.

2) **Quadratic functions.** These are functions of the form: $\boldsymbol{y = a\,x^2 + b\,x + c}$**,**

where a, b and c are constants. Their graphs are called **parabolas**. Falling objects move along parabolic paths. If a is a positive number then the parabola opens upward and if a is a negative number then the parabola opens downward.

3) **Power functions.** These are functions of the form: $\boldsymbol{y = a\,x^b}$,

where a and b are constants. They get their name from the fact that the variable x is raised to some power.

4) **Exponential functions.** These are functions of the form: $\boldsymbol{y = a\,b^x}$,

where x is in an exponent (not in the base as was the case for power functions) and a and b are constants. (Note that only b is raised to the power x; not a.)

If the base b is greater than 1 then the result is exponential growth. Many physical quantities grow exponentially (e.g. animal populations).

If the base b is smaller than 1 then the result is exponential decay. Many quantities decay exponentially (e.g. the sunlight reaching a given depth of the ocean and the speed of an object slowing down due to friction).

5) **Logarithmic functions.** These are functions of the form:

$\mathbf{y = \log_b x}$ (the logarithm of x to base b), where b is not equal to 1, and x is positive.

The logarithm of a number is the exponent to which another fixed number, the *base*, must be raised to produce that number: x = $\boldsymbol{b^y}$

- The logarithm to base 10 (that is b = 10) is called the **common logarithm** y = log x
- The logarithm to base 2 (that is b = 2) is called the **binary logarithm** $y = \log_2 x$
- The logarithm to base e (that is b = e) is called the **natural logarithm** y = ln x

where e ≈ 2.718 and is the limit of $\left(1+\frac{1}{n}\right)^n$ as n approaches infinity

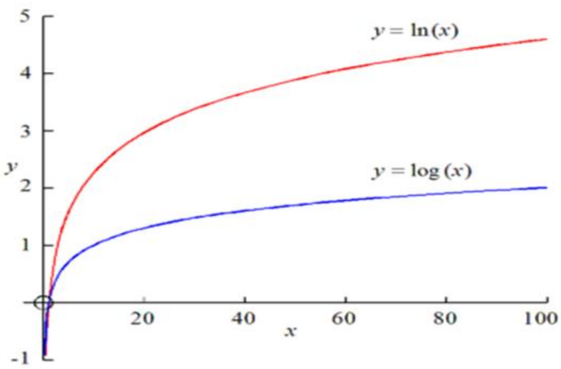
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{or} \quad e = \sum_{n=0}^{\infty}\frac{1}{n!}$$

- $\lim\limits_{x\to x_\circ} f(x) = L$: we say that $f(x)$ approaches the limit as x approaches $x_\circ$

  $\lim\limits_{x\to 2} f(x) = 2$: the limit of f of x as x tends to one is equal to two

- $\sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \ldots$ sum of $\frac{1}{n!}$ for i from nought [zero] to infinity

  sum over i from zero to infinity of $\frac{1}{n!}$



the graph gets arbitrarily close to the y axis, but does not meet or intersect it.

6) **Sinusoidal functions.** These are functions of the form: ***y* = *a* sin (*b x* + *c*),** where *a*, *b* and *c* are constants. Sinusoidal functions are useful for describing anything that has a wave shape with respect to position or time.

1. Derivative of a function

2.1 The **derivative** of a function is the instantaneous rate of change of a function.

The derivative of a function y = f(x) of a variable x is a measure of the rate at which the value y of the function changes with respect to the change of the variable x. It is called the derivative of f with respect to x.

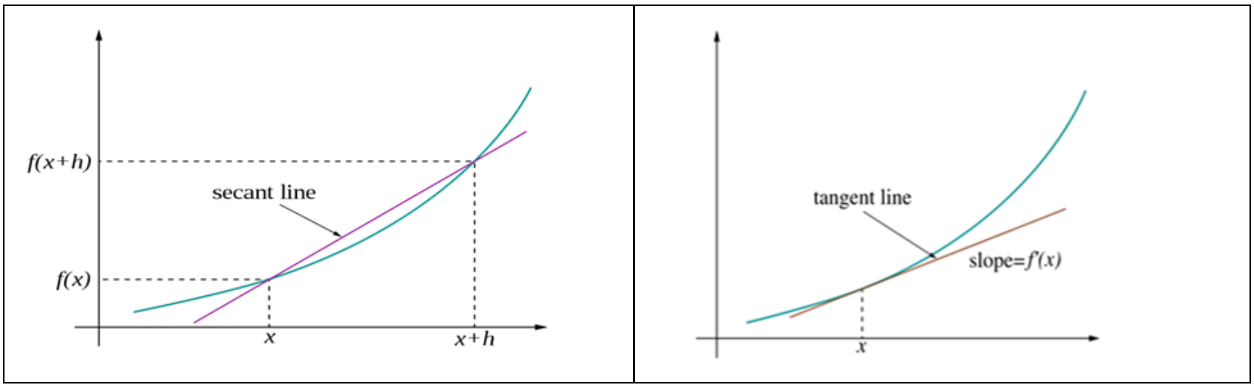
The derivative of f (x) with respect to x is the function f′(x) and is defined as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df}{dx} = \frac{dy}{dx}$$

the limit of the difference quotient (the change in f over the change in x) as Δx approaches zero.

The **derivative** of a function measures the sensitivity to change of the function value (output value) with respect to a change in its argument (input value).

The derivative of a function of a single variable at a chosen input value, when it exists, is the slope of the tangent line to the graph of the function at that point.



That is, as the distance between the two x points (h = Δx) becomes closer to zero, the slope of the line between them becomes closer to resembling a tangent line.

2.2 Partial derivative

partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant (as opposed to the total derivative, in which all variables are allowed to vary)

Exemple:
If the cylinder has a radius r and length (height) h, then its volume is given by $V = \pi r^2 h$
The surface area of a **closed cylinder** is made up the sum of all three components: top, bottom and side. Its surface area is
$S = \pi r^2 + \pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi rh = 2\pi r(r + h) = \pi d(r + h)$, where d is the diameter.

a ***partial derivative*** of the volume with respect to height represents the rate with which the volume changes if its height is varied and its radius is kept constant.

3. Differential of a function: *the principal part of the change in a function y = f(x) with respect to changes in the independent variabl*e (*variabl*es).

$$df = f'(x)\,dx$$

1.1 In the case of function of several variables, such that $f = f(x,y,z,..) = f(x_1, x_2,.. x_n)$

$$df = f'_x dx + f'_y dy + \ldots = \sum_{i=1}^{n} f'_{x_i} dx_i$$

and is called the total differential

1.2 Applications of the total differential.

1.2.1 Error estimations: $\Delta f \approx df = f'_x dx + f'_y dy + \ldots$

Ex.1 Let f is $S = 4\pi R^2$, *and* $R = 1.1 \pm 0.1$

Ex.2 Let f is $S = xy$, *and* $x = 2.1 \pm 0.1$; $y = 1.3 \pm 0.2$

1.2.2 Approximate computations:

- linear approximation $\Delta f \approx df$
- Taylor series approximation

a Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point

$$\Delta f \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Ex.1.: Estimate absolute and relative change in volume of a cylinder with respect to change in height and radius $V = \pi r^2 h,\ r = r_0 + \Delta r,\ h = h_0 + \Delta h$

II. Test Problems

1. Find the value of the function at the given points:

$F(x,y) = x^3 - 3xy - y^2 + 2x + y$; $x=4$, $y=3$

2. Compute the limit:

$\lim\limits_{x\to 2}\left(3x^2+7x-1\right) =$

3. Find derivatives of the following functions:

$y = (x+1)\cdot\left(x^3+3\right)$

$y = \dfrac{3+x^3}{x+1}$

$y = (\sin x)\cdot(\cos x)$

$y = (\sin x)\cdot(\ln x)$

$y = \ln(\sin x)$

$y = \sin(\ln x)$

$y = e^{\sin x}$

$y = \sin e^{x^2}$

4. Find partial derivatives of the given functions with respect to each variable:

$f = x^2 + y^2 + z^2$

$f = \ln(x+y+z)$

$f = e^{x+y+z}$

5. Find a total differential of the function:

$f = x\cdot y\cdot z$

6. Find the absolute and relative increase in cylinder volume $V = \pi r^2 h$, with radius of the base $r = 1$cm, and length (hight) $l = 10$ cm, if the change in radius stands at $\Delta r = 0.1$ cm, change in length (hight) $\Delta l = 0.2$ cm.

5. Find the integrals:

$\int(5+3\sin x-2\cos x)dx =$

$\int e^{x+1}dx =$

6. Find the area of figures bounded by:

$f(x) = \frac{1}{x^2}$, $x = 1$, $x = 2$, $y = 0$ (axis X)

$f(x) = e^x$, $x = 1$, $x = \ln 3$, $y = 0$ (axis X)

7. Find the average value of functions on the given intervals [a;b]:

$f(x) = x^2$, $a = 1$, $b = 3$

$f(x) = \sin x$, $a = 0$, $b = 2\pi$

Reading:

- Medical and Biological Physics. Textbook for students of higher medical education institutions / V.I. Fediv, O.I. Olar, V.V. Kulchynskyi, H. Y. Rudko. – Chernivtsi: BSMU Publishing House, 2017. - 342 p. – English. pp. 68 – 99.

Watching:

https://www.youtube.com/watch?v=-ktrtzYVk_I

https://www.youtube.com/watch?v=QqF3i1pnyzU&list=RDCMUCRGXV1QlxZ8aucmE45tRx8w&index=2

https://www.youtube.com/watch?v=e1nxhJQyLYI&list=RDCMUCRGXV1QlxZ8aucmE45tRx8w&index=3

https://www.youtube.com/watch?v=8B31SAk1nD8&list=RDCMUCRGXV1QlxZ8aucmE45tRx8w&index=2

https://www.youtube.com/watch?v=cyi-qyG1Yds&list=RDCMUCRGXV1QlxZ8aucmE45tRx8w&index=1

https://www.youtube.com/watch?v=KKg88oSUv0o&list=RDCMUCRGXV1QlxZ8aucmE45tRx8w&index=4